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Flow past a sphere undergoing unsteady rectilinear motion and unsteady drag at small Reynolds number

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The flow induced by a sphere which undergoes unsteady motion in a Newtonian fluid at small Reynolds number is considered at distances large compared with sphere radius a. Previous solutions of the unsteady Oseen equations (Ockendon 1968; Lovalenti & Brady 1993*b*) for rectilinear motion are refined. Three-dimensional Fourier transforms of the disturbance field are integrated over Fourier space to derive new concise equations for the velocity field and history force in terms of single history integrals.

Various slip-velocity profiles are classified by the ratio A of the particle relative displacement, $z'_p(t') - z'_p(\tau')$, to the diffusion length, $l'_D = 2[\nu(t' - \tau')]^{1/2}$, where ν is the kinematic viscosity of the fluid. Most previous studies are concerned with large-displacement motions for which the ratio is large in the long-time limit. It is shown using asymptotic calculations that the flow at any point at large distance z past a sphere for arbitrary large-displacement and non-reversing motion is the same as the steady-state laminar wake if z is expressed in terms of the time elapsed since the particle was at that point in the laboratory frame. The point source solution for the remainder of the far flow is also valid for the unsteady case.

A start-up motion with slip velocity $V'_p = \gamma'(t')^{-1/2}$, $t' > 0$, is investigated for which A is finite. A self-similar solution for the flow field is obtained in terms of space coordinates scaled by the diffusion length, $\boldsymbol{u}' = a\boldsymbol{u}^{ss}(\boldsymbol{\eta})/t'$ where $\boldsymbol{\eta} = \boldsymbol{r}'/2(\nu t')^{1/2}$. The unsteady Oseen correction to the drag is inversely proportional to time.

When A is small in the long-time limit (a small-displacement motion) the flow field also depends on the space coordinates in terms of $\boldsymbol{\eta}$. The distribution of the streamwise velocity u_z is symmetrical in z.

1. Introduction

The disturbance flow at distances of order a from a sphere of radius a and a small particle Reynolds number, $Re = aU_c/\nu$, based on a characteristic particle velocity U_c is governed to the leading order by the creeping-flow equations. Their solution gives the Stokes drag on the sphere. At distances comparable with the Oseen length, $l'_O = \nu/U_c \gg a$, the inertial terms in the Navier–Stokes equations become of the same order as the viscous ones. When the time scale of particle velocity variation is of the order of the Oseen time scale, $t'_O = \nu/U_c^2$, the unsteadiness of the flow should also be taken into account. As a result the disturbance velocity field in the Oseen region differs from the Stokeslet field. The first-order force on the sphere also differs from the classical Basset history force.

The particle induces only a small disturbance to the background flow at distances $r' \gg a$. This enables us to linearize the inertial terms. The particle effect may be approximated by a point force. Then the dimensionless momentum equation for a sphere undergoing arbitrary unsteady rectilinear motion in unbounded fluid being at rest at infinity can be written as (Childress 1964)

$$\frac{\partial \boldsymbol{u}}{\partial t} - V_p \frac{\partial \boldsymbol{u}}{\partial z} + \nabla p - \nabla^2 \boldsymbol{u} = 6\pi V_p \boldsymbol{e}_z \delta(\boldsymbol{r}). \tag{1.1}$$

Here the particle $V_p(t)$ and the fluid $\boldsymbol{u}(t,\boldsymbol{r})$ velocities are scaled by U_c and $Re\,U_c$ respectively, the space coordinates by l'_O, and time by t'_O. The z-axis is aligned with the direction of the motion, $\boldsymbol{e}_z$ is a unit vector along the z-axis, and the origin of the coordinate system is at the centre of the sphere and translates with the particle velocity V_p.

Ockendon (1968) studied unsteady rectilinear particle motion. He used the three-dimensional Fourier transform of the disturbance field

$$\left\{ \begin{array}{c} \boldsymbol{\Gamma} \\ \Pi \end{array} \right\} = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} \left\{ \begin{array}{c} \boldsymbol{u} \\ p \end{array} \right\} \exp\left(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}\right) \mathrm{d}\boldsymbol{r}$$

to solve (1.1) together with the continuity equation. As a result the system is reduced to an ordinary differential equation for the Fourier transform of the disturbance velocity. Its general solution is

$$\boldsymbol{\Gamma} = -\frac{3}{4\pi^2} \int_{-\infty}^{t} \exp[-k^2 \Delta\tau + \mathrm{i}k_z \Delta z_p(\tau)] \left(\frac{k_z \boldsymbol{k}}{k^2} - \boldsymbol{e}_z \right) V_p(\tau) \mathrm{d}\tau, \tag{1.2}$$

$$k = |\boldsymbol{k}|, \quad \Delta\tau = t - \tau, \quad \Delta z_p = z_p(t) - z_p(\tau), \quad z_p(t) = \int_{-\infty}^{t} V_p(\tau)\,\mathrm{d}\tau.$$

Here $z_p(t)$ is the displacement of the particle. Then the velocity field is given by the inverse Fourier transform,

$$\boldsymbol{u} = \iiint_{-\infty}^{\infty} \boldsymbol{\Gamma} \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}\right) \mathrm{d}\boldsymbol{k}. \tag{1.3}$$

Two of the integrations in (1.3) were performed using a spherical coordinate system (r, θ, φ), and the streamfunction of the axisymmetric flow was presented as the double integral

$$\psi = \frac{3r^2 \sin^2\theta}{2\pi^{1/2}} \int_0^t \frac{V_p(\tau)}{r_1^2} \left[\int_0^1 \exp\left(-\frac{r_1^2 s^2}{4\Delta\tau} \right) \mathrm{d}s - \exp\left(-\frac{r_1^2}{4\Delta\tau} \right) \right] \frac{\mathrm{d}\tau}{\Delta\tau^{1/2}}, \tag{1.4}$$

where $r_1^2 = r^2 + 2z\Delta z_p + \Delta z_p^2$. The velocity field yields for the drag on the sphere scaled by $\mu a U_c$

$$F = -6\pi \left[V_p(t) + \frac{Re}{\pi^{1/2}} \left\{ \int_{-\infty}^{t} \frac{\mathrm{d}V_p}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\Delta\tau^{1/2}} \right.\right.$$
$$\left.\left. -3 \int_0^1 \int_0^{z_p(t)} (\exp(-A^2 s^2) - \exp(-A^2) + (s^2 - 1)A^2) \frac{\mathrm{d}\Delta z_p}{\Delta z_p^2 \Delta\tau^{1/2}} \mathrm{d}s \right\} \right]$$
$$+ O(Re^2 \ln Re), \tag{1.5}$$

where

$$A = \frac{\Delta z_p}{2\Delta\tau^{1/2}}. \tag{1.6}$$

Here the leading-order term is a quasi-steady Stokes drag. Terms of order Re are divided into the classical Basset force (the first term in curly brackets) and the extra history force. For the motion with the characteristic time t'_c such that $a^2/\nu \ll t'_c \ll t'_O$ it can be shown that the first term is dominant, i.e. the history force matches with the Basset solution. However when $t'_c \sim t'_O$ the above breakdown is more conventional since the origin of both terms is the same. They originate at large distances from the sphere and behave as the homogeneous part of the $O(Re)$ solution at the particle centre.

Lovalenti & Brady (1993*b*) extended the analysis to an arbitrary time-dependent particle velocity and to particles of arbitrary shape when the history force has not only the longitudinal (drag) but also a transversal component. They did not solve for the detailed velocity field but used the reciprocal theorem to compute the force on the particle. For rectilinear particle motion their expression for the unsteady Oseen force is

$$F_O = -3\pi^{1/2} Re \int_{-\infty}^{t} \left\{ V_p(t) - \frac{3V_p(\tau)}{2A^2} \left[\frac{\pi^{1/2}}{2A} \mathrm{erf}\,(A) - \exp\left(-A^2\right) \right] \right\} \frac{\mathrm{d}\tau}{\Delta\tau^{3/2}}. \tag{1.7}$$

One of the main objects of the present work is to refine the analysis for the rectilinear motion. The new expressions for the velocity field and the history force, simpler than (1.4), (1.5) and (1.7), will be derived in terms of the single history integrals.

Equation (1.4) describes the diffusion and convection of the vorticity introduced into the fluid at different instants τ. The diffusion length can be defined as a function of elapsed time, $l_D(\Delta\tau) = 2\Delta\tau^{1/2}$, or in dimensional form, $l'_D = 2[\nu(t'-\tau')]^{1/2}$. The function $A(t,\tau)$ entering into both equations (1.5) and (1.7) for the unsteady Oseen drag is the ratio of the particle relative displacement to the diffusion length. It characterizes the relationship between convection and diffusion processes for the momentum released into the fluid at the time τ.

The types of unsteady motion can be classified according to the magnitude of A in the long-time limit, $\Delta\tau \to \infty$. When the relative displacement is large compared to the diffusion length, $A \gg 1$, convection dominates in the streamwise direction in the far field. The balance of the streamwise convection and the cross-stream diffusion generates a laminar wake downstream of the sphere. The field in the remainder of the flow is a symmetric point mass source.

The various slip-velocity profiles for the large-displacement motions were studied to understand the flow pattern in the far field and the behaviour of the unsteady Oseen force in the long-time limit. Bentwich & Miloh (1978) and Sano (1981) considered the start-up motion with constant velocity, $V_p(t) = 1$ as $t > 0$. Bentwich & Miloh (1978) obtained the velocity field in the Oseen region using a Laplace transform in time of the governing equations. Sano (1981) calculated the force on the sphere for this type of motion:

$$\begin{aligned} F = -6\pi \Bigg\{ 1 + \frac{\delta(t)}{3} + \frac{3}{8} Re \Bigg[\left(1 + \frac{4}{t^2}\right) \mathrm{erf}\left(\frac{t^{1/2}}{2}\right) \\ + \frac{2}{(\pi t)^{1/2}} \left(1 - \frac{2}{t}\right) \exp\left(-\frac{t}{4}\right) \Bigg] + \frac{9}{40} Re^2 \log Re + O\left(Re^2\right) \Bigg\}. \end{aligned} \tag{1.8}$$

The unsteady Oseen force tends to a steady-state value, and the difference decays like t^{-2} as $t \gg 1$.

Small oscillations in the free-stream velocity (Mei, Lawrence & Adrian 1991, Mei & Adrian 1992; Lovalenti & Brady 1993*a*), and a step change in the particle velocity (Lovalenti & Brady 1993*b*) have also been considered. The force approaches the steady Oseen force differently for the various types of slip velocity: like t^{-1}, t^{-2} or exponential decay as $t \gg 1$. Hinch (1993) explained the different behaviours of the history force in the long-time limit by the change in the wake past a sphere. Many recent works have investigated the history forces at finite Reynolds numbers by including $O(Re^2)$ terms. A review of the literature on the subject can be found in Michaelides (1997).

Large-displacement motion with the familiar pattern in the far field is not the only possibility. There are velocity profiles for which the displacement remains finite for all times. Then the ratio of the relative displacement to the diffusion length is small in the long-time limit. A particle freely decelerating in a quiescent fluid is an example of small-displacement motion. For times large compared with the relaxation time of the particle velocity the diffusion of the momentum dominates in the far region. Another example existing in practice is an oscillating particle with $V_p = \cos(\omega t)$. This profile arises when a wave propagates in a carrier fluid.

Of interest is whether there exists a type of motion for which A remains finite for all times. Since the diffusion length is proportional to the square root of time one has to assume a power-law profile of the slip velocity, $z_p \sim t^{1/2}$, $V_p \sim t^{-1/2}$, to obtain $A \sim 1$ as $t \to \infty$. To eliminate a singularity of V_p at $t \to +0$ one can consider a profile with finite velocity during some small time interval $(0, t_1)$, $t_1 \ll t$ and at a later time obeying the power law. Such a particle velocity profile is unlikely to be observed in practice, like many other prescribed profiles studied earlier. However it is worth investigating the disturbance field for $V_p \sim t^{-1/2}$ since it is a transitional regime between the two broad classes, the large-displacement and small-displacement motions. Convection and diffusion balance over the entire far region. A self-similar solution can be obtained for the disturbance flow. This solution can be a good test for the numerical simulations of unsteady flows past a small sphere.

The plan of the work as follows. In §§2 and 3 new concise expressions for the disturbance field and the history force respectively are derived. Fourier transform (1.2) is integrated over Fourier space to give the velocity field and the unsteady Oseen force as single history integrals. In §4 it is shown by asymptotic calculation of the history velocity field that the steady-state laminar wake when expressed in terms of elapsed time remains valid for arbitrary large-displacement and non-reversing motion. The self-similar flow field is obtained in §5 for the slip velocity $V_p = \gamma t^{-1/2}$. It is valid over the entire Oseen region, $r' \gg a$ and for all times $t > 0$. The velocity field and Oseen force for small-displacement motion in the long-time limit are considered in §6. The streamwise velocity is symmetrical in z. Three examples of small-displacement motion are considered: flow after a short-term motion; start-up motion with power-law slip velocity, $V_p(t) = t^{-m}$, $1/2 < m < 1$; and an oscillating sphere, $V_p = \cos(\omega t)$. Results are summarized in §7.

2. Unsteady velocity field

Our study of the rectilinear unsteady motion starts from Ockendon's solution (1.2) for the Fourier transform of the disturbance velocity. The streamwise component of

Fourier transform (1.2) can be transformed using integration by parts to give

$$\begin{aligned}
\Gamma_z &= \frac{3}{4\pi^2}\int_{-\infty}^{t} \exp\left(-k^2\Delta\tau + \mathrm{i}k_z\Delta z_p\right)\frac{k_x^2+k_y^2}{k^2}\mathrm{d}z_p(\tau) \\
&= \frac{3}{4\pi^2}\int_{-\infty}^{t} \exp\left[-k^2\Delta\tau + \mathrm{i}k_z z_p(t)\right]\frac{\mathrm{i}\left(k_x^2+k_y^2\right)}{k_z k^2}\mathrm{d}\exp\left[-\mathrm{i}k_z z_p(\tau)\right] \\
&= \frac{3}{4\pi^2}\left[\frac{\mathrm{i}\left(k_x^2+k_y^2\right)}{k_z k^2} - \int_{-\infty}^{t}\exp\left(-k^2\Delta\tau + \mathrm{i}k_z\Delta z_p\right)\frac{\mathrm{i}\left(k_x^2+k_y^2\right)}{k_z}\mathrm{d}\tau\right] \\
&= \frac{3}{4\pi^2}\int_{-\infty}^{t}\exp\left(-k^2\Delta\tau\right)\frac{\mathrm{i}\left(k_x^2+k_y^2\right)}{k_z}E\left(k_z,\tau\right)\,\mathrm{d}\tau, \qquad (2.1)
\end{aligned}$$

where

$$E = 1 - \exp(\mathrm{i}k_z\Delta z_p) = 1 - \exp\left[\mathrm{i}k_z\int_{\tau}^{t} V_p(s)\,\mathrm{d}s\right].$$

The last expression for Γ_z is convenient for performing integration of the inverse Fourier integral (1.3) over k_x and k_y. Then the z-component of the velocity field can be written as

$$\begin{aligned}
u_z &= \frac{3}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{t}\frac{\mathrm{i}}{k_z}\exp\left(\mathrm{i}k_z z - k_z^2\Delta\tau - \frac{R^2}{4\Delta\tau}\right) \\
&\quad\times\left(\frac{1}{\Delta\tau^2} - \frac{R^2}{4\Delta\tau^3}\right)E(k_z,\tau)\,\mathrm{d}\tau\mathrm{d}k_z \qquad (2.2a) \\
&= \frac{3}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{t}\frac{\mathrm{i}}{k_z}\exp(\mathrm{i}k_z z - k_z^2\Delta\tau)\,E(k_z,\tau)\,\mathrm{d}\exp\frac{(-R^2/4\Delta\tau)}{\Delta\tau}\,\mathrm{d}k_z \qquad (2.2b) \\
&= \frac{3}{4\pi}\mathrm{i}\int_{-\infty}^{\infty}\int_{-\infty}^{t}\frac{1}{\Delta\tau}\exp\left(\mathrm{i}k_z z - k_z^2\Delta\tau - \frac{R^2}{4\Delta\tau}\right) \\
&\quad\times\left[k_z E(k_z,\tau) + \mathrm{i}V_p(\tau)\exp\left(\mathrm{i}k_z\Delta z_p\right)\right]\mathrm{d}\tau\mathrm{d}k_z. \qquad (2.2c)
\end{aligned}$$

where $R^2 = x^2 + y^2$. The equation (2.2*c*) is obtained from (2.2*b*) using integration by parts over τ.

Finally, integration over k_z permits us to present u_z as a single history integral:

$$\begin{aligned}
u_z &= \frac{3}{8\pi^{1/2}}\int_{-\infty}^{t} Q(z,R,t,\tau)\,\mathrm{d}\tau \qquad (2.3a) \\
&= \frac{3}{8\pi^{1/2}}\int_{-\infty}^{t}\left[2V_p(\tau)\Delta\tau - z - \Delta z_p\right]\exp\left(-\frac{(z+\Delta z_p)^2 + R^2}{4\Delta\tau}\right) \\
&\quad\times\frac{\mathrm{d}\tau}{\Delta\tau^{5/2}} + \frac{3}{2}\frac{z}{r^3}, \qquad (2.3b)
\end{aligned}$$

where

$$Q = \frac{1}{\Delta\tau^{5/2}}\left\{\left[2V_p(\tau)\Delta\tau - z - \Delta z_p\right]\exp\left(-\frac{(z+\Delta z_p)^2 + R^2}{4\Delta\tau}\right) + z\exp\left(-\frac{r^2}{4\Delta\tau}\right)\right\}. \qquad (2.4)$$

When $r \ll 1$ the velocity field obtained matches with the quasi-steady Stokeslet,

$$u_{zS} = \frac{3}{4} V_p(t) \left(\frac{1}{r} + \frac{z^2}{r^3} \right).$$

It is obvious from physical reasoning that the flow field near the particle is due to its motion during a small time interval near $\tau = t$. This result also follows from (2.3) if one assumes that $z \sim R \sim r \ll 1$. The exponents in (2.4) are not small only for $\Delta\tau_S \sim r^2 \ll 1$, and integration in (2.3) over this small interval gives the Stokeslet singularity at $r \ll 1$. Taking into account that $\Delta z_p = V_p(t)\Delta\tau + O(\Delta\tau^2)$ when $\Delta\tau \ll 1$, one can present the integrand of (2.3a) as a series in $\Delta\tau$:

$$Q = \frac{1}{\Delta\tau^{5/2}} \left[(V_p(t)\Delta\tau - z) \exp\left(-\frac{z V_p(t)}{2} \right) + z + O\left(\Delta\tau^{3/2}\right) \right] \exp\left(-\frac{r^2}{4\Delta\tau} \right)$$

$$= Q_S + O(\Delta\tau^{-1}) \quad \text{for} \quad \Delta\tau \sim r^2 \ll 1,$$

where

$$Q_S = V_p(t) \left(\frac{1}{\Delta\tau^{3/2}} + \frac{z^2}{2\Delta\tau^{5/2}} \right) \exp\left(-\frac{r^2}{4\Delta\tau} \right).$$

Integration of Q_S over τ yields the Stokeslet field u_{zS}, and the difference

$$u_z - u_{zS} = \frac{3}{8\pi^{1/2}} \int_{-\infty}^{t} (Q - Q_S)\, \mathrm{d}\tau, \tag{2.5}$$

is regular as $r \to 0$.

Solution (2.3) for the longitudinal velocity enables us also to find a radial component of axisymmetric flow, or, equivalently, a streamfunction of axisymmetric flow. It is convenient to use the cylindrical coordinate system, (z, R, ϕ), for which

$$u_R = -\frac{1}{R}\frac{\partial\psi}{\partial z}, \quad u_z = \frac{1}{R}\frac{\partial\psi}{\partial R}.$$

The last equation can be solved to give, taking account of (2.3a),

$$\psi = \int_0^R R_1 u_z(z, R_1)\, \mathrm{d}R_1 = \frac{3}{4\pi^{1/2}} \int_{-\infty}^{t} Q\Delta\tau \left[\exp\left(\frac{R^2}{4\Delta\tau} \right) - 1 \right] \mathrm{d}\tau. \tag{2.6}$$

A steady Oseen field for the motion with constant velocity, $V_p(t) = 1$, $\Delta z_p = \Delta\tau$, can also be derived from equations (2.3) and (2.6). Performing the integration in (2.3b) and (2.6) one obtains

$$u_z^{sO} = \frac{3}{4r} \left[\frac{2z}{r^2} + \exp\left(-\frac{r+z}{2} \right) \left(1 - z\frac{2+r}{r^2} \right) \right], \tag{2.7}$$

$$\psi^{sO} = \frac{3}{2} \left(1 - \frac{z}{r} \right) \left[1 - \exp\left(-\frac{r+z}{2} \right) \right]. \tag{2.8}$$

The above equations are consistent with the Oseen velocity field (Landau & Lifshitz 1986, p. 96) and the streamfunction (Van Dyke 1964) written in a spherical coordinate system.

The type of particle motion frequently considered is a start-up problem, when $V_p = 0$ for $t < 0$, $z_p(0) = 0$. Equation (2.3) can be applied to obtain a solution at $r' \gg a$, $t' \gg a^2/\nu$, or in dimensionless form $r \gg Re$, $t \gg Re^2$, in this case. The particle does not induce disturbances in the fluid at $t < 0$. However the integrand of (2.3a)

is not zero for $\tau < 0$. Since the particle is at rest so that $\Delta z_p = z_p(t)$ when $\tau < 0$ the integral over this interval can be calculated explicitly. As a result the velocity field can be rewritten as

$$u_z = \frac{3}{8\pi^{1/2}} \int_0^t Q \mathrm{d}\tau + \frac{3}{t}[\zeta_0 h(\eta_0) - \zeta h(\eta)], \tag{2.9}$$

where Q is given by (2.4) and

$$\left.\begin{aligned} h(x) &= \frac{1}{4\pi^{1/2}x^2}\exp(-x^2) - \frac{1}{8x^3}\mathrm{erf}(x), \\ \zeta &= \frac{z}{2t^{1/2}}, \quad \eta = \frac{r}{2t^{1/2}}, \quad \xi = \frac{R}{2t^{1/2}}, \\ \zeta_0 &= \frac{z+z_p(t)}{2t^{1/2}} = \zeta + \zeta_p, \quad \eta_0 = \left[\frac{(z+z_p(t))^2+R^2}{4t}\right]^{1/2} \\ &\qquad = (\zeta_0^2+\xi^2)^{1/2}, \quad \zeta_p = \frac{z_p(t)}{2t^{1/2}}. \end{aligned}\right\} \tag{2.10}$$

Thus the velocity field for the start-up motion involves, in addition to a history term, the difference of the same function of two distances: from the particle position and from the point $(0, 0, -z_p(t))$ where the motion started (in the laboratory frame). Hinch (1993) and Mei & Lawrence (1996) derived similar behaviour for the far field, $r \gg t^{1/2}$, by considering the wake for the start-up problem with constant velocity. There is a mass deficit in the wake. The mass is supplied by a point mass source of flow rate 6π at the particle centre. The mass sink of the same flow rate is at the starting point where the wake terminates. As a result the velocity is the superposition of two spherically symmetric flows:

$$u_z = \frac{3}{2}\left\{\frac{z}{r^3} - \frac{z+z_p}{\left[(z+z_p(t))^2+R^2\right]^{3/2}}\right\} \quad \text{for} \quad r \gg t^{1/2}. \tag{2.11}$$

Equation (2.9) is the generalization of this result for distances comparable with the diffusion length and arbitrary start-up motion. At large r and arbitrary $V_p(t)$ equation (2.11) can be readily deduced from (2.3) and (2.9). For any point far from the particle path, or equivalently, when $(z + \Delta z_p)^2 + R^2 \gg t$ for any τ, all the exponents, both in the integrand of (2.9) and in the equation (2.10), are asymptotically small. Then the main contribution to the far flow gives the function h which can be approximated by

$$h(x) \approx -\frac{1}{8x^3} \quad \text{as} \quad x \to +\infty, \tag{2.12}$$

and the velocity field (2.9) is reduced to (2.11).

The streamfunction for the start-up motion can be written, taking account of (2.6) and (2.9), as

$$\begin{aligned} \psi = \frac{3}{2}\Bigg\{ & \frac{1}{2\pi^{1/2}} \int_0^t Q\Delta\tau \left[\exp\left(\frac{R^2}{4\Delta\tau}\right) - 1\right] \mathrm{d}\tau \\ & + \frac{\zeta_0 \mathrm{erf}(\eta_0)}{\eta_0} - \mathrm{sgn}(\zeta_0)\mathrm{erf}(|\zeta_0|) + \mathrm{sgn}(\zeta)\mathrm{erf}(|\zeta|) - \frac{\zeta\mathrm{erf}(\eta)}{\eta}\Bigg\}. \end{aligned} \tag{2.13}$$

The non-integral terms in the solution (2.9), (2.13) for the start-up motion depend on the new dimensionless coordinates ζ, η and ξ, the distance from the starting point η_0 and particle displacement ζ_p. All of them are scaled by diffusion length, $l_D(t) = 2t^{1/2}$, or in dimensional form, $l'_D = 2(\nu t')^{1/2}$. They describe the transport of momentum generated at $t = 0$ when the slip velocity changes impulsively. For general unsteady motion the particle introduces vorticity into the fluid continuously, and the solution for the velocity field (2.3) accounts for the history of momentum transport. Hence one should define the diffusion length more generally as a function of time interval, $l_D(\Delta\tau) = 2(t-\tau)^{1/2}$. The function $A(t,\tau)$ given by (1.6) represents the ratio of the relative particle displacement and the diffusion length. It characterizes the relationship between the convection and diffusion processes for different time intervals. It is small for small $\Delta\tau$, and this time interval gives the velocity field at small distances from the sphere where the diffusion dominates.

For distances $r \sim 1$ the exponent in the integrand of (2.3) is finite for $\Delta\tau$ of order unity or larger. The field in the Oseen region is due to integration over $\Delta\tau \sim 1$ since the contribution of large intervals decays as $\Delta\tau^{-3/2}$. Function A is of order unity for this time interval, and this means that convection and diffusion balance in the region. The field in the far region, $r \gg 1$, originates from the long-time history, $\Delta\tau \gg 1$, since large intervals only enables the exponent in the integrand of (2.3) to be finite. We consider various cases, for which $A \gg 1$, $A \sim 1$ or $A \ll 1$ when $\Delta\tau \gg 1$, in §§4, 5 and 6 respectively.

3. Unsteady Oseen force

For motion with characteristic time of order t'_O the Oseen force comes from the homogeneous part of the disturbance velocity at the particle position. One need only determine $(u_z - u_{zS})$ at $r = 0$ and use the Stokes drag law. In view of (2.5) the dimensionless force is

$$F_O = \frac{9\pi^{1/2}}{4} Re \int_{-\infty}^{t} \{[2V_p(\tau)\Delta\tau - \Delta z_p]\exp(-A^2) - V_p(t)\Delta\tau\}\frac{\mathrm{d}\tau}{\Delta\tau^{5/2}}. \tag{3.1}$$

Taking into account that

$$\frac{\mathrm{d}\exp(-A^2)}{\mathrm{d}\tau} = \exp(-A^2)A\frac{2V_p(\tau)\Delta\tau - \Delta z_p}{2\Delta\tau^{3/2}},$$

one can obtain another concise form of the equation for the history force:

$$F_O = \frac{9\pi^{1/2}}{4} Re \int_{-\infty}^{t} \left[\frac{4\Delta\tau}{\Delta z_p}\frac{\mathrm{d}\exp(-A^2)}{\mathrm{d}\tau} - V_p(t)\right]\frac{\mathrm{d}\tau}{\Delta\tau^{3/2}}.$$

For the start-up motion this equation can be rewritten in view of (2.9) as

$$F_O = \frac{9\pi^{1/2}}{4} Re \left\{\int_{0}^{t} \{[2V_p(\tau)\Delta\tau - \Delta z_p]\exp(-A^2) - V_p(t)\Delta\tau\} \times\frac{\mathrm{d}\tau}{\Delta\tau^{5/2}} - \frac{2V_p(t)}{t^{1/2}} + \frac{8\pi^{1/2}\zeta_p}{t}h(\zeta_p)\right\} \tag{3.2a}$$

$$= \frac{9\pi^{1/2}}{4} Re \left\{\int_{0}^{t} \left[\frac{4\Delta\tau}{\Delta z_p}\frac{\mathrm{d}\exp(-A^2)}{\mathrm{d}\tau} - V_p(t)\right]\frac{\mathrm{d}\tau}{\Delta\tau^{3/2}} - \frac{2V_p(t)}{t^{1/2}} + \frac{8\pi^{1/2}\zeta_p}{t}h(\zeta_p)\right\}. \tag{3.2b}$$

Unfortunately, the author is not able to prove the equivalence of the new unsteady Oseen force and the previous expressions (1.5) and (1.7). However all known solutions for the force for the special cases of the slip-velocity profile (small oscillations in the free-stream velocity, a step change in the particle velocity) can also be deduced from equations (3.1), (3.2). So, for the start-up motion with constant velocity, $V_p(t) = 1$ when $t > 0$, one has $\Delta z_p = \Delta\tau$, $A = \Delta\tau^{1/2}/2$, $\zeta_p = t^{1/2}/2$. Performing the integration in (3.2*a*) one obtains Sano's (1981) result (equation (1.8)).

4. Flow field in the far region for large-displacement motion

In this section we consider the behaviour of the unsteady flow following from (2.3) at distances large compared with the Oseen length, $r \gg 1$, and for the long-time limit, $t \gg 1$, for the case when the particle displacement is large, so that it satisfies the condition $A(t, \tau) \gg 1$, or equivalently,

$$|z_p(t) - z_p(\tau)| \gg (t-\tau)^{1/2} \quad \text{for} \quad t-\tau \gg 1, \tag{4.1}$$

for all instants τ.

A particle in a steady motion, $V_p(t) = 1$, obviously satisfies this condition. There are well-known solutions for the far field (Landau & Lifshitz 1986), the laminar wake past the sphere,

$$u_z = -\frac{3}{2z}\exp\left(\frac{R^2}{4z}\right), \quad R^2 \sim |z| \gg 1, \;\; z < 0,$$

and the point source solution in the remainder of the flow,

$$u_z = \frac{3}{2}\frac{z}{r^3}. \tag{4.2}$$

The wake flow can also be written in terms of the time required to translate from the particle position to z, or equivalently, the time elapsed since the particle was at z (in the laboratory frame), namely $t = -z$:

$$u_z = \frac{3}{2t}\exp\left(-\frac{R^2}{4t}\right). \tag{4.3}$$

The main features of the large-displacement unsteady flow in the far region are similar to those for the steady case. The far-flow solutions (4.2) and (4.3) remain valid for a non-reversing particle motion, $V_p(t) > 0$, satisfying (4.1).

The wake field can be derived by an asymptotic estimation of (2.3) for a negative and large z. For the unsteady non-reversing motion each $z < 0$ can be attributed to the instant $\tau = \tau_e(z)$ when the particle was at z. This time can be found from the equation

$$z_p(\tau_e) = z + z_p(t).$$

The exponent in the history velocity field (2.3) accounts for the diffusion during the elapsed time $\Delta\tau_e = t - \tau_e$ of the momentum released at z. A relatively small time interval close to instant $\tau = \tau_e$ (of the order of $\Delta\tau_e^{1/2} \ll \Delta\tau_e$) contributes to the disturbance velocity at $z < 0$, $|z| \gg 1$, since the translation of the momentum source for the large-displacement motion is fast compared to the diffusion rate in the long-time limit, $\Delta\tau_e \gg 1$. This follows also from (2.3*b*). The integrand is not exponentially small when

$$|\Delta z_p + z| \sim \Delta\tau_e^{1/2} \ll \Delta\tau_e \sim |z|, \quad R \sim \Delta\tau_e^{1/2}.$$

The elapsed time $\Delta\tau$ for this small region (compared to $|z|$) is nearly constant, $\Delta\tau = \Delta\tau_e[1 + O(\Delta\tau_e^{-1/2})]$, and the pre-exponent factor in the first term of (2.4) can be estimated as

$$\frac{2V_p(\tau)\Delta\tau - z - \Delta z_p}{\Delta\tau^{5/2}} = \frac{2V_p(\tau)}{\Delta\tau_e^{3/2}}[1 + O(\Delta\tau_e^{-1/2})].$$

This enables the integral (2.3b) to be calculated to the leading order of $\Delta\tau_e^{-1/2}$:

$$u_z = \frac{3}{4\pi^{1/2}}\int_{-\infty}^{t}\exp\left[-\frac{(z+\Delta z_p)^2 + R^2}{4\Delta\tau_e}\right]\frac{V_p(\tau)\,\mathrm{d}\tau}{\Delta\tau_e^{3/2}}$$

$$= \frac{3}{4\pi^{1/2}}\int \exp\left[-\frac{\left(z+\Delta z_p\right)^2 + R^2}{4\Delta\tau_e}\right]\frac{\mathrm{d}[-(z+\Delta z_p)]}{\Delta\tau_e^{3/2}}$$

$$= \frac{3}{2\Delta\tau_e}\exp\left(-\frac{R^2}{4\Delta\tau_e}\right) \quad \text{for} \quad R \sim \Delta\tau_e^{1/2},\ z < 0. \tag{4.4}$$

Thus the expression for the velocity distribution within the wake in terms of the elapsed time for arbitrary rectilinear non-reversing motion is the same as (4.3) obtained for the steady case. It depends on $\Delta\tau_e$ only but involves neither the distance from the particle z nor even the particle velocity at the instant τ_e. The reason is that the momentum the particle releases due to Stokes drag into the fluid per unit time is $6\pi V_p(\tau_e)$ at small Reynolds number. Hence the momentum released per unit length of the wake which is $V_p(\tau_e)$ times less does not depend on the particle velocity.

Hinch (1993) used similar arguments to explain the temporal behaviour of the unsteady Oseen force for various types of motion: an impulsive start or stop, a step change in the slip velocity. For the step increase of velocity from V_o to V_n the old and new wakes have the same width $t^{1/2}$ at $z = -V_n t$ where they join and the same momentum sources per unit length. The difference between the unsteady Oseen force and the steady-state value is due to the vorticity diffusing from the transition zone back to the particle and it decays exponentially with time. Mei & Lawrence (1996) studied the flow field for a step change in the slip velocity at finite Reynolds numbers and the effect in the wake due to the difference in the steady-wake volume fluxes. For the case when the Reynolds number is small and the two fluxes are equal their results for both the old and new wakes and the transition zone are the same as (4.4).

The potential source flow can also be derived from (2.3b) for the unsteady case when $r \gg 1$, $R^2 \gg |z|$ or $z \gg 1$. The integrand of (2.3b) then becomes exponentially small for all τ since $\Delta z_p \gg \Delta\tau^{1/2}$, and the velocity equals the last term in (2.3b), or is the same as (4.2).

5. Self-similar solution for the disturbance field

The start-up problem with $V_p(t) \sim t^{-1/2}$ is an intermediate regime between the large-displacement and small-displacement motions. The function $A(t,\tau)$ is of order unity for this profile, and the convection and the diffusion balance over the entire far region. A self-similar solution is obtained for the disturbance flow.

The particle and the fluid are quiescent for $t < 0$, and for $t > 0$ the dimensional velocity is

$$V_p' = \gamma'(t')^{-1/2} \quad \text{for} \quad t > 0. \tag{5.1}$$

The functional form of the self-similar flow can be deduced from a dimensional analysis. The dimension of the constant γ' in (5.1) is $[lt^{-1/2}]$ or the square root of the dimension $[l^2t^{-1}]$ of the kinematic viscosity ν. One could not introduce the characteristic velocity U_c (nor the Reynolds number, the Oseen length and the time, l'_O and t'_O) using a combination of the two dimensional values only: γ' and ν. So one has to scale the velocity field and to introduce the Reynolds number using the time t' and the diffusion length $l'_D = 2(t'\nu)^{1/2}$ as the characteristic time and length scales. The dimensionless solution may then depend on the space coordinates only in terms of the combination

$$\boldsymbol{\eta} = \boldsymbol{r}'/2(t'\nu)^{1/2} = (\zeta, \xi, \phi).$$

Thus, the dimensional disturbance velocity must be of the form

$$u_z'^{ss} = \frac{a}{t'} u_z^{ss}(\gamma, \boldsymbol{\eta}),$$

where γ is a single dimensionless group defined by $\gamma = \gamma'\nu^{-1/2}$. It characterizes the relative size of convection and diffusion. When γ is small diffusion dominates while for large γ convection is dominant, and the main features of the flow are similar to the steady Oseen solution.

Dependence $u_z^{ss}(\gamma, \boldsymbol{\eta})$ can be found from (2.9). One can use some arbitrary but constant characteristic velocity U_c and characteristic time $t'_O = \nu/U_c^2$ to non-dimensionalize the governing equations. Irrespective of the choice of U_c the dimensionless particle velocity and the displacement are

$$V_p^{ss} = \gamma t^{-1/2}, \quad z_p^{ss} = 2\gamma t^{1/2}, \tag{5.2}$$

and the displacement scaled by the diffusion length is constant, $\zeta_p^{ss} = \gamma$. Then the integrand of (2.9) can be presented as

$$\left.\begin{aligned} Q^{ss} &= t^{-2} q, \\ q &= \frac{2}{\Delta s^{5/2}} \left\{ (\gamma s^{-1/2}\Delta s - \Delta\zeta_p^{ss} - \zeta) \exp\left(-\frac{(\zeta + \Delta\zeta_p^{ss})^2 + \xi^2}{\Delta s} \right) + \zeta \exp\left(-\frac{\eta^2}{\Delta s} \right) \right\}, \\ s &= \frac{\tau}{t}, \quad \Delta s = \frac{\Delta\tau}{t} = 1 - s, \quad \Delta\zeta_p^{ss} = \frac{\Delta z_p}{2t^{1/2}} = \gamma(1 - s^{1/2}). \end{aligned}\right\} \tag{5.3}$$

As a result we have from (2.9) for the velocity field:

$$u_z = t^{-1} u_z^{ss}(\gamma, \boldsymbol{\eta}), \tag{5.4}$$

$$u_z^{ss} = 3\left[\frac{1}{8\pi^{1/2}} \int_0^1 q \,\mathrm{d}s + \zeta_0^{ss} h(\eta_0^{ss}) - \zeta h(\eta) \right], \tag{5.5}$$

$$\zeta_0^{ss} = \zeta + \gamma, \quad \eta_0^{ss} = (\zeta_0^{ss2} + \xi^2)^{1/2}.$$

Some comments should be made about the particle velocity as $t \to +0$. The velocity profile (5.1) tends to infinity at small t, and the above analysis ceases to be valid when the Reynolds number becomes finite. To eliminate the infinite values of $V_p(t)$ one can assume that the particle has some finite velocity during a small time interval, $V_p(\tau) = O(t_1^{-1/2})$ for $0 < \tau < t_1$, $t_1 = o(t)$, and at a later time the slip velocity obeys the power law (5.2) (see figure 1). To ensure $Re \ll 1$ during this interval the dimensional time should satisfy the inequality $t'_1 \gg (\gamma' a/\nu)^2 = \gamma^2 a^2/\nu$.

The integrand of the history equation for the velocity field (2.3*b*) for such a slip-velocity profile differs from the self-similar value over the small interval $(0, t_1)$ only,

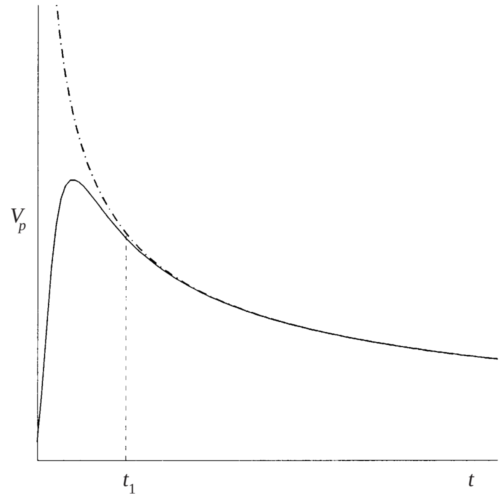

FIGURE 1. The disturbance field is self-similar for the particle velocity $V_p^{ss} = \gamma t^{-1/2}$ (dashed–dotted line). The solid line is the finite dependence differing from V_p^{ss} during small time interval $(0, t_1)$. The field for this velocity profile equals, to the leading order of t_1/t, the self-similar solution.

and the velocity field can be presented as

$$\begin{aligned} u_z &= \frac{3}{8\pi^{1/2}}\left(\int_0^{t_1} Q\mathrm{d}\tau + \int_{t_1}^{t} Q^{ss}\mathrm{d}\tau\right) + 3t^{-1}[\zeta_0 h(\eta_0) - \zeta h(\eta)] \\ &= t^{-1}u_z^{ss} + \frac{3}{8\pi^{1/2}}\int_0^{t_1}(Q - Q^{ss})\mathrm{d}\tau + 3t^{-1}[\zeta_0 h(\eta_0) - \zeta_0^{ss} h(\eta_0^{ss})]. \end{aligned} \quad (5.6)$$

Since the particle displacement differs only slightly from the power law,

$$z_p(t) = \int_{t_1}^{t} V_p^{ss}(\tau)\,\mathrm{d}\tau + \int_0^{t_1} V_p(\tau)\,\mathrm{d}\tau = z_p^{ss}\left\{1 + O\left[(t_1/t)^{1/2}\right]\right\} \quad \text{for} \quad t_1 \ll t, \quad (5.7)$$

$$\zeta_0 = \zeta_0^{ss}\left\{1 + O\left[(t_1/t)^{1/2}\right]\right\}, \quad \eta_0 = \eta_0^{ss}\left\{1 + O\left[(t_1/t)^{1/2}\right]\right\},$$

the last term in (5.6) is much smaller, of the order of $(t_1/t)^{1/2}$, than the self-similar solution u_z^{ss}. Similarly, the relative displacement is close to the self-similar value,

$$\left.\begin{aligned} \Delta z_p = \int_{\tau}^{t} V_p(\tau)\,\mathrm{d}\tau = z_p^{ss}\{1 + O[(t_1/t)^{1/2}]\}, \\ \Delta\tau = t[1 + O(t_1/t)] \quad \text{for} \quad \tau \sim t_1. \end{aligned}\right\} \quad (5.8)$$

Hence the leading-order term of the integrand of (5.6) is due to the velocity difference, which is finite over the interval $(0, t_1)$:

$$Q - Q^{ss} \simeq 2(V_p(\tau) - V_p^{ss})\exp(-\eta_0^{ss2})t^{-3/2}.$$

As a result the integral

$$\int_0^{t_1}(Q - Q^{ss})\,\mathrm{d}\tau \sim (t_1/t)^{1/2}t^{-1} \ll t^{-1} \quad \text{for} \quad t_1/t \ll 1,$$

and is also small compared with $t^{-1}u_z^{ss}$. Therefore, the disturbance velocity for the above dependence $V_p(t)$ is the same as the self-similar solution with the accuracy

$(t_1/t)^{1/2}$. The history forces for the two profiles will be identical with the same accuracy.

The self-similar flow has no characteristic time and has no long-time limit. Its behaviour remains the same for all times. It should be stressed that the solution (5.5) could not be treated as the long-time limit of the flow with characteristic time t_1. The value of t_1 and the specific velocity profile within the interval $(0, t_1)$ can be chosen arbitrarily. Hence t_1 is not a relevant time scale for the flow at $t \gg t_1$ and is introduced to eliminate the singularity at $t \to +0$ only.

A similar problem occurs for the step variation of the slip velocity. This profile implies an infinite particle acceleration and infinite force at $t = 0$. One has to introduce some finite interval $(0, t_1)$ when the particle velocity varies from one constant value to another. However this transient process has not been studied, since the time of velocity variation is assumed to be small compared with the Oseen time t'_O and also is not a relevant time scale for the flow at $r' \sim l'_O$, $t' \sim t'_O$.

The streamfunction for the self-similar solution, in view of (2.13), is

$$\psi = \psi^{ss}(\gamma, \boldsymbol{\eta}), \tag{5.9}$$

$$\psi^{ss} = \frac{3}{2}\left\{\frac{1}{2\pi^{1/2}}\int_0^1 q\Delta s\left[\exp\left(\frac{\xi^2}{\Delta s}\right) - 1\right]\mathrm{d}s + \frac{\zeta_0^{ss}\mathrm{erf}\,(\eta_0^{ss})}{\eta_0^{ss}} - \mathrm{sgn}\,(\zeta_0^{ss})\,\mathrm{erf}\,(|\zeta_0^{ss}|)\right.$$
$$\left. + \mathrm{sgn}\,(\zeta)\mathrm{erf}\,(|\zeta|) - \frac{\zeta\,\mathrm{erf}\,(\boldsymbol{\eta})}{\boldsymbol{\eta}}\right\}.$$

The self-similar flow given by (5.4) expands equally in all directions as $t^{1/2}$ with the magnitude of the disturbance decaying as t^{-1}. The flow within the wake exhibits some similar features. The wake considered in the laboratory frame also grows diffusively as $t^{1/2}$, and the velocity disturbance decays as t^{-1}. However the expansion occurs within a narrow region downstream the sphere in the normal direction only. Besides, the wake-flow solution is obtained for distances large compared with the Oseen length while the self-similar solution (5.4) is valid over the entire Oseen region, $r' \gg a$.

The particle displacement (or equivalently, the translation of the momentum source) is comparable with the diffusion length for any time for the velocity profile $V_p^{ss} = \gamma t^{-1/2}$ as $\gamma \sim 1$. As a result the exponent in the history velocity field (2.3*b*) which accounts for the convection and diffusion of momentum is of the order unity for all times. Thus the contributions of the various preceding instants to the self-similar velocity field are of the same order, in contrast to the wake flow which is influenced by a small vicinity of the elapsed time only.

The two flows have the same disturbance decay but different volumes of the disturbed regions. This difference can be explained from momentum conservation arguments. For the wake flow, convection is dominant in the far field, and the velocity field is governed by a dimensionless momentum released per unit length which is constant, 6π. As a result the disturbance is inversely proportional to the square of the wake width, $\delta^{-2} = (t^{1/2})^{-2} = t^{-1}$. The self-similar flow is governed by the total momentum, $6\pi\int_0^t V_p^{ss}(\tau)\,\mathrm{d}\tau = 6\pi z_p^{ss}(t)$, which grows as $t^{1/2}$. Hence the disturbance is proportional to the momentum divided by the volume of disturbed region $6\pi z_p^{ss}(t)(t^{1/2})^{-3}$, i.e. it also decays as t^{-1}.

The results of calculations of the self-similar streamwise velocity and the streamfunction fields using (5.5) and (5.10) are shown in figures 2 and 3 respectively. To compare the flow patterns we also present in figure 4 the fields for the steady Oseen solution, $u_z^{sO}(z, R)$ and $\psi_z^{sO}(z, R)$, given by (2.7) and (2.8).

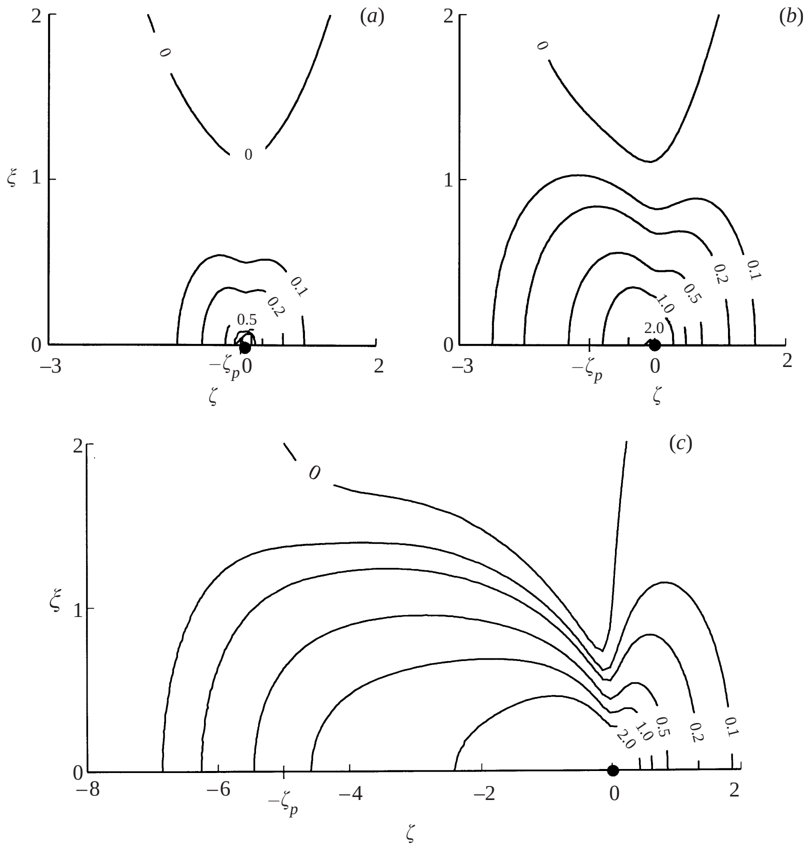

FIGURE 2. Self-similar disturbance field u_z^{ss} for the slip-velocity profile $V_p^{ss} = \gamma t^{-1/2}$: (*a*) $\gamma = 0.2$, (*b*) $\gamma = 1$, (*c*) $\gamma = 5$. Velocity u_z^{ss} is nearly symmetric in ζ at small γ when the starting point $(-\zeta_p)$ is close to the particle position. At large γ the starting point is far from the particle position, and the flow pattern is similar to steady Oseen flow at $\xi, \zeta \sim 1$.

The self-similar streamwise velocity is nearly symmetric in ζ at small γ when the starting point is close to the particle position (see figure 2*a*). This flow pattern has similar features to those for the small-displacement motion considered in § 6 (cf. figures 2*a* and 7). The disturbed region becomes more elongated in the streamwise direction as γ grows. At large γ when the starting point is far from the particle position the flow pattern is similar to the steady Oseen flow (cf. figures 2*c* and 4*a*, 3*c* and 4*b*). The reason is that convection dominates at distances $\eta \sim 1$, and the flow is quasi-steady.

The force on the sphere for a slip-velocity profile, to $O(Re)$, is

$$F = -6\pi\gamma t^{-1/2} + Re\, t^{-1} f_O^{ss}(\gamma).$$

The Oseen correction $f_O^{ss}(\gamma)$, taking account of (3.2*a*) and (5.3), can be presented as

$$f_O^{ss} = \frac{9\pi^{1/2}}{2}\left\{\int_0^1 \left[(\gamma s^{-1/2}\Delta s - \Delta\zeta_p^{ss})\exp\left(-\frac{(\Delta\zeta_p^{ss})^2}{\Delta s}\right) - \frac{\gamma\Delta s}{2}\right]\frac{\mathrm{d}s}{\Delta s^{5/2}} - \gamma + 4\pi^{1/2}\gamma h(\gamma)\right\}. \quad (5.10)$$

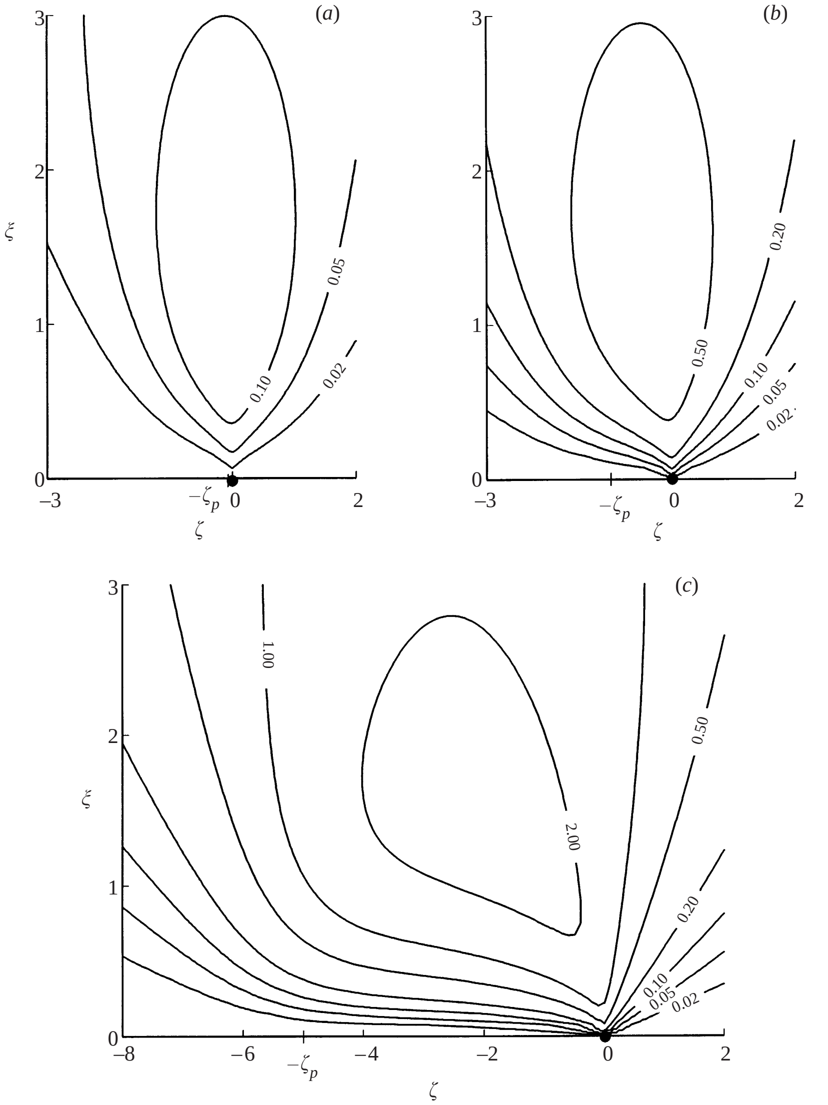

FIGURE 3. Self-similar streamfunction field ψ^{ss}. The values of γ are the same as in figure 2.

The contributions of the various s or the various instants of time to the Oseen force (5.10) are of the same order as $\gamma \sim 1$. For this reason it differs for all times from the quasi-steady Oseen correction which is also inversely proportional to the time,

$$F_O^{qs} = -\frac{9\pi}{4} Re\, V_p^2(t) = -\frac{9\pi}{4} Re\, \gamma^2 t^{-1}. \tag{5.11}$$

The unsteady Oseen force for large-displacement motion, in contrast to above case with $A \sim 1$, is due to the integration over $\Delta\tau = O(1)$ since for large elapsed times $A \gg 1$ and their contribution is exponentially small.

The dimensions analysis states that an $O(Re)$ force must decay like t^{-1} regardless the approach used to solve the governing equations. For example if we rewrite the Sano solution (1.8) in dimensional form and replace V_p by $\gamma t^{-1/2}$ we obtain such a decay. One would expect that the Basset history force also decays as t^{-1}.

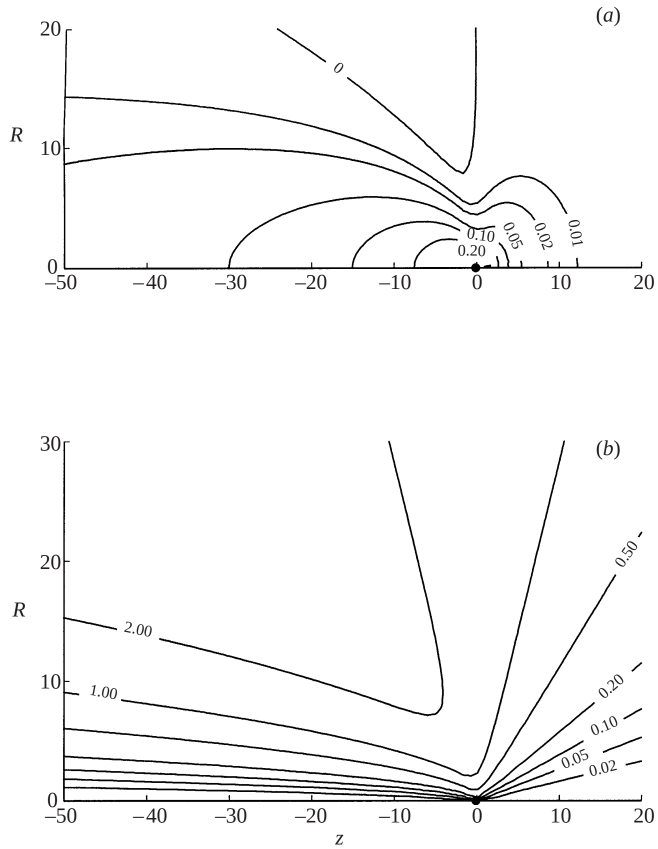

FIGURE 4. (a) Streamwise velocity and (b) streamfunction for the steady Oseen solution.

However the Basset integral is divergent as $t \to +0$ for the power slip velocity. The reason is that the solution of the unsteady Stokes equations does not account for the displacement of momentum source due to particle translation and assumes that the momentum is released at the same point. The problem can be eliminated only for the profile with finite velocity shown in figure 1. The contribution of the small interval $(0, t_1)$ to the Basset integral can be estimated as $(t_1 t)^{-1/2}$. This value is large compared not only with the estimate $O(t_1^{1/2} t^{-3/2})$ following from (5.7), (5.8) for this interval but even with the contribution $O(t^{-1})$ of the interval (t_1, t). Thus the Basset formula predicts even the scaling of the history force for the power profile incorrectly since it states that the small initial interval yields the dominant contribution to the force.

For the limiting cases of very small and very large values of γ the analytical solution for the Oseen force can be obtained from (5.10). When $\gamma \ll 1$ the flow is nearly symmetric in ζ (see figure 2a), and the force is small. The exponent in the integrand of (5.10) and the function h can be expanded as a series of γ:

$$\exp\left(-\frac{(\Delta\zeta_p^{ss})^2}{\Delta s}\right) = 1 - \frac{1 - s^{1/2}}{1 + s^{1/2}}\gamma^2 + O(\gamma^4),$$

$$h = \pi^{-1/2}\left(-\tfrac{1}{6} + \tfrac{1}{10}\gamma^2\right) + O(\gamma^4) \quad \text{for} \quad \gamma \ll 1.$$

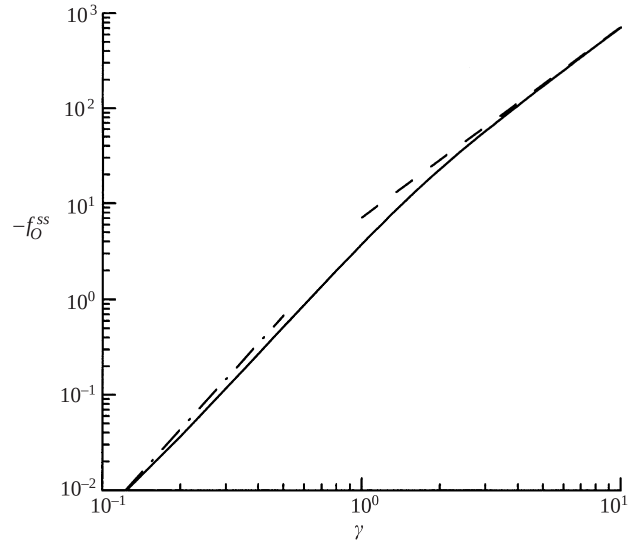

FIGURE 5. Oseen correction to the force on a particle for the slip-velocity profile $V_p^{ss} = \gamma t^{-1/2}$. At small γ it grows as γ^3 (dashed–dotted line, equation (5.12)) and at large γ it is close to quasi-steady Oseen correction (dashed line, equation (5.13)).

The integral can be presented in terms of a γ-series as

$$\gamma \int_0^1 \left(\frac{2 + s^{1/2}}{2} - \frac{\gamma^2}{1 + s^{1/2}} \right) \frac{\mathrm{d}s}{s^{1/2}(1 + s^{1/2})^{5/2} (1 - s^{1/2})^{1/2}} = \gamma \frac{5}{3} - \gamma^3 \left(\frac{7}{8} + \frac{3\sqrt{2}\pi}{64} \right).$$

As a result the terms of order γ in (5.10) cancel out, and the leading-order solution for the Oseen correction is of order γ^3:

$$f_O^{ss} = -\gamma^3 \tfrac{9}{640} \pi^{1/2} (152 + 15\sqrt{2}\pi) \quad \text{for} \quad \gamma \ll 1. \tag{5.12}$$

For the opposite limiting case of large γ the dominant contribution to the integral comes from the small interval $\Delta s \sim \gamma^{-2}$ where the exponent in the integrand of (5.10) is not small. For such a small time interval the particle velocity variation can be neglected, $V_p = \gamma[1 + O(\gamma^{-2})]$. This means that the flow near the particle looks quasi-steady. Changing the variables in (5.10) using $x = \Delta s \gamma^2$, $s = 1 - x\gamma^{-2}$, one can obtain to the leading order

$$f_O^{ss} = -\frac{9\pi^{1/2}}{4} \gamma^2 \int_0^\infty \left[1 - \exp\left(-\frac{x}{4} \right) \right] \frac{\mathrm{d}x}{x^{3/2}} = -\frac{9\pi}{4} \gamma^2 \quad \text{for} \quad \gamma \gg 1. \tag{5.13}$$

Thus the force at large γ is close to the quasi-steady Oseen value (5.11). Figure 5 shows the dependence of the Oseen correction f_O^{ss} on γ calculated with the use of (5.10). The quasi-steady solution overestimates the unsteady Oseen force.

6. Small-displacement motion

In this section we consider the behaviour of the far field in the long-time limit for small-displacement motion when the relative particle displacement is small compared with the diffusion length, $A(t, \tau) \ll 1$, or equivalently,

$$|\Delta z_p| = |z_p(t) - z_p(\tau)| \ll (t - \tau)^{1/2} \quad \text{for} \quad t - \tau \gg 1, \tag{6.1}$$

for any instant τ. In this case the momentum transport at a distance $\eta \sim 1$ is due to the diffusion. The velocity field (2.3a) subject to the condition (6.1) can be simplified. The first exponent in (2.4) can be expanded as

$$\exp\left(-\frac{(z+\Delta z_p)^2+R^2}{4\Delta\tau}\right) = \exp\left(-\frac{r^2}{4\Delta\tau}\right)\left[1-\frac{z\Delta z_p}{2\Delta\tau}+O(\Delta z_p^2 t^{-1})\right],$$

and the integrand of (2.3a) becomes

$$Q = \frac{1}{\Delta\tau^{5/2}}\left[2V_p(\tau)\Delta\tau - \Delta z_p + \frac{z^2\Delta z_p}{2\Delta\tau}\right][1+O(\Delta z_p^2\Delta\tau^{-1})]$$
$$\times\exp\left(-\frac{r^2}{4\Delta\tau}\right) \quad \text{for} \quad \Delta\tau \gg 1,\ r\sim\Delta\tau^{1/2}. \tag{6.2}$$

It follows from (6.2) that the leading-order velocity field in the long-time limit can be presented in terms of ζ, η as

$$u_z = g_1(\eta,t) + \zeta^2 g_2(\eta,t) \quad \text{for} \quad t\gg 1,\ \eta\sim 1, \tag{6.3}$$

$$g_1 = \frac{3}{4\pi^{1/2}t}\int_{-\infty}^{1}[t^{1/2}V_p(s)\Delta s - \Delta\zeta_p]\exp\left(-\frac{\eta^2}{\Delta s}\right)\frac{ds}{\Delta s^{5/2}},$$

$$g_2 = \frac{3}{2\pi^{1/2}t}\int_{-\infty}^{1}\Delta\zeta_p\exp\left(-\frac{\eta^2}{\Delta s}\right)\frac{ds}{\Delta s^{7/2}}.$$

Thus the distribution of the streamwise velocity u_z in the far field is symmetric in ζ for the small-displacement motion.

For the start-up motion satisfying condition (6.1) the distance between the starting point and the particle position scaled by the diffusion length is small, $\zeta_p = z_p(t)/2t^{1/2} \ll 1$. As a result the two distances, η_0 and η, are close,

$$\eta_0 = \eta + \frac{\zeta}{\eta}\zeta_p + O(\zeta_p^2),$$

so that the second term in (2.9) can be expanded as

$$\frac{3}{t}[\zeta_0 h(\eta_0) - \zeta h(\eta)] = \frac{3\zeta_p}{t}\left[h(\eta) + \frac{\zeta^2}{\eta}h'(\eta) + O(\zeta_p)\right],$$

where $h'(\eta)$ denotes the derivative of the function h. Thus the functional form of u_z remains the same as (6.3) for the start-up motion with the functions g_1 and g_2 given by

$$g_1 = \frac{3}{t}\left\{\frac{1}{4\pi^{1/2}}\int_0^1[t^{1/2}V_p(s)\Delta s - \Delta\zeta_p]\exp\left(-\frac{\eta^2}{\Delta s}\right)\frac{ds}{\Delta s^{5/2}} + \zeta_p h(\eta)\right\}, \tag{6.4a}$$

$$g_2 = \frac{3}{t}\left[\frac{1}{2\pi^{1/2}}\int_0^1\Delta\zeta_p\exp\left(-\frac{\eta^2}{\Delta s}\right)\frac{ds}{\Delta s^{7/2}} + \zeta_p\frac{h'(\eta)}{\eta}\right]. \tag{6.4b}$$

Several types of particle velocity profile satisfying (6.1) are considered below: motion during a small time interval; an oscillating sphere; the start-up motion with power-law slip velocity $V_p(t) = t^{-m}$, $1/2 < m < 1$.

6.1. *Flow after a short-term motion*

The particle moves with an arbitrary velocity $V_p(\tau) \sim 1$ in a time interval $0 < \tau < t_2$, $t_2 \sim 1$. We study the velocity field at a later time such that $t \gg t_2$. The displacement remains the same after the particle stops, $z_p(t) = z_p(t_2) \sim t_2$ for $t > t_2$, and the integration in (6.4) should be performed over a small interval $(0, s_2)$, $s_2 = t_2/t \ll 1$.

The first term in square brackets in (6.4*a*) is large compared with the second one, $t^{1/2} V_p(s)\Delta s \sim t^{1/2} \gg \Delta\zeta_p \sim s_2 t^{1/2}$. Taking into account that the elapsed time is close to t,

$$\Delta\tau = t[1 + O(t_2/t)], \quad \Delta s \equiv 1 - \tau/t = 1 + O(t_2/t) \quad \text{for} \quad 0 < \tau < t_2,$$

we have from (6.4*a*)

$$g_1 = \frac{3}{4(\pi t)^{1/2}} \exp(-\eta^2) \int_0^{s_2} V_p(s)\, \mathrm{d}s + \frac{3\zeta_p}{t} h(\eta)$$

$$= \frac{3z_p(t_2)}{16 t^{3/2}} [\pi^{-1/2}(4 + 2\eta^{-2}) \exp(-\eta^2) - \eta^{-3} \operatorname{erf}(\eta)]. \tag{6.5}$$

The term involving the history integral in square brackets in equation (6.4*b*) is of the order of s_2^2, or much less than the last term, which is of order s_2. As a result the function g_2 to the leading order of s_2 is

$$g_2 = \frac{3\zeta_p}{t} \frac{h'(\eta)}{\eta} = \frac{3z_p(t_2)}{2t^{3/2}} \frac{h'(\eta)}{\eta}. \tag{6.6}$$

Therefore, the velocity field for this case depends only on the total displacement of the particle, $z_p(t_2)$, but not the history of the motion.

At distances large compared with the diffusion length, $\eta \gg 1$, the flow is inviscid. In view of (2.12) the derivative in equation (6.6) can be approximated by $h'(\eta) = 3/8\eta^4$ as $\eta \to +\infty$. As a result the solution obtained reduces to

$$u_z = \frac{3z_p(t_2)}{16t^{3/2}} \left(-\frac{1}{\eta^3} + \frac{3\zeta^2}{\eta^5} \right) = \frac{3z_p(t_2)}{2r^3} \left(-1 + \frac{3z^2}{r^2} \right) \quad \text{for} \quad r \gg t^{1/2}.$$

This field corresponds to a stationary potential dipole flow. It is due to a close mass source and sink, at the particle position and the starting point respectively. The dipole strength is the product of a dimensionless flow rate 6π and the total displacement $z_p(t_2)$.

The unsteady Oseen force can be readily calculated using (3.2*a*). The integrand is non-zero when $0 < \tau < t_2$ only, and the elapsed time $\Delta\tau$ is close to t over this interval. Taking into account that $A \ll 1$ and $\zeta_p = z_p(t_2)/2t^{1/2} \ll 1$ the force to the leading-order of t^{-1} can be written as

$$F_O = \frac{9\pi^{1/2}}{4} Re \left[t^{-3/2} \int_0^t 2V_p(\tau)\, \mathrm{d}\tau + \frac{8\pi^{1/2}\zeta_p}{t} h(\zeta_p) \right] = Re \frac{3\pi^{1/2} z_p(t_2)}{t^{3/2}}. \tag{6.7}$$

Thus the Oseen force for the slip-velocity profile considered also depends on the total displacement only. It should be stressed that the problem considered differs from the stopping problem studied by Lovalenti & Brady (1993*b* and Hinch (1993). In that case the particle translates with constant velocity at $\tau < 0$, i.e. the total particle displacement and the momentum introduced into the fluid are infinite. As a result a slower decay of the history force, like t^{-1}, was obtained.

The solution (6.5) is applicable also for a start-up motion of a heavy particle freely decelerating in a fluid. Such a situation can arise when the particle experiences an impulsive force or crosses a shock wave in a gas. The dimensionless particle velocity scaled by $V_p'(0)$ and displacement, to the leading order of Re, are

$$V_p = \exp(-t/t_r), \quad z_p = t_r[1 - \exp(-t/t_r)] \quad \text{for} \quad t > 0,$$

where $t_r = m_p/6\pi\mu a$ is the relaxation time of the particle velocity, m_p is particle mass. The total particle displacement is also finite in this case. The above speculation remains valid at large times, $t \gg t_r$, since the times $\tau \sim t_r$ only give the finite contribution to the integral in (6.4a).

The solution for the disturbance velocity enables us to find the particle velocity to $O(Re)$ terms, i.e. the velocity due to the disturbances induced at early stages of motion. The homogeneous part of the fluid velocity at the particle position is

$$(u_z - u_{zS})|_{r\to 0} = \frac{t_r}{2\pi^{1/2}t^{3/2}} \quad \text{for} \quad t \gg t_r.$$

As a result the two-term solution for the velocity of the freely decelerating sphere is given by

$$V_p = \exp(-t/t_r) + Re\,\frac{t_r}{2\pi^{1/2}t^{3/2}} \quad \text{for} \quad t \gg t_r.$$

Note that this equation is valid regardless of the relation between Re and t/t_r. So the first term, which decays faster with time, can be smaller than the second one.

6.2. *Oscillating particle*

Another interesting example of particle motion satisfying condition (6.1) is an oscillating motion. The slip velocity profile and particle displacement are given by

$$V_p = \cos(\omega t), \quad z_p = \omega^{-1}\sin\omega t. \tag{6.8}$$

Mie (1994) studied the disturbance flow and the force on the sphere for the velocity profile $V_p = \exp(\mathrm{i}\omega t)$ at finite Reynolds numbers using a finite-difference solution of the governing equations and a Fourier transform in the frequency domain. He was specially interested in the case when $\omega \ll 1$ and the amplitude of the oscillation is large compared with the Oseen length. The flow field at small Reynolds number is divided into three regions. The flows within an inner Stokes region and intermediate Oseen region are quasi-steady. The flow within a third region with $z \sim \omega^{-1} \gg R \sim \omega^{-1/2}$ is similar to the unsteady wake.

The drag was presented in terms of a Fourier transform in the frequency domain as

$$F = Re\sum_{n=1}^{N} D_{2n-1}\exp[\mathrm{i}(2n-1)\omega t].$$

Mei retained first three terms in the sum. The numerical solution showed that the imaginary part of the Fourier components D_{2n-1}, $n = 1, 2, 3$ grows as $\omega\ln\omega$ in the low-frequency limit.

It is of interest to compare Mei's numerical result with that following from (3.1). For the profile considered, $V_p = \cos(\omega t)$, the Oseen force should be decomposed into an even and an odd part. Then the odd part corresponds to the imaginary part of the drag calculated by Mei.

A time interval of order unity only contributes to the drag since the exponent in the integrand of (3.1) is not small for $\Delta\tau \sim 1$. For these times the particle velocity

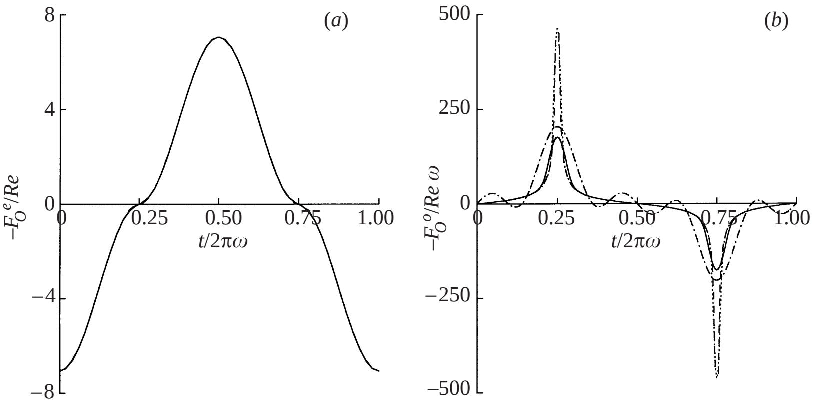

FIGURE 6. Oseen force on the oscillating sphere at small ω. The force is decomposed into (*a*) even and (*b*) odd parts. Solid and dotted lines are the results of the numerical calculation of the history force (3.1) for $\omega = 10^{-3}$ and 2×10^{-5} respectively. The dashed lines are the asymptotic solution (6.9). The dashed–dotted line is Mei's (1994) numerical result for $\omega = 2 \times 10^{-5}$ based on the Fourier transform of the flow in the time domain with only three terms retained. The asymptotic prediction of the even part differs slightly from the numerical results near the cuspidal points only, $t = \pi/2$, $3\pi/2$. It agrees well with the numerical values of the odd part also except the vicinity of the cuspidal points. A few terms of the Fourier transform is not sufficient to approximate the odd part.

and the relative displacement can be expanded in terms of an ω-series as

$$V_p(\tau) = \cos(\omega t - \omega\Delta\tau) = \cos(\omega t) + \omega\Delta\tau \sin(\omega t) + O(\omega^2),$$

$$\Delta z_p = \omega\Delta\tau\cos(\omega t) + \frac{(\omega\Delta\tau)^2}{2}\sin(\omega t) + O(\omega^3) \quad \text{for} \quad \omega \ll 1,\ \Delta\tau \sim 1.$$

As a result one can obtain from (3.1) the two-term expansion of the Oseen force:

$$F_O = \frac{9\pi}{4} Re \left[-\cos(\omega t)|\cos(\omega t)| + 2\omega \frac{\sin(\omega t)}{|\cos(\omega t)|} \right] \quad \text{for} \quad \omega \ll 1. \qquad (6.9)$$

Thus the unsteady force is close to the quasi-steady Oseen force (the leading-order term). The reason is that the particle velocity varies slowly compared with the Oseen time. A perturbation of the quasi-steady solution (the first-order term) is due to the small particle acceleration. However the deviation from the quasi-steady value becomes of the same order, and the solution in terms of an ω-series ceases to be valid when the particle is close to the cuspidal points, $t_n = \pi(1/2 + n)/\omega$. In the vicinity of the points the particle velocity and the first term in (6.9) tend to zero while the second one tends to infinity. The size of the region and the scaling of the Oseen force can be estimated by equating the orders of the two terms. As a result we have $\cos^3[\omega(t - t_n)] = O(\omega)$ and $F_O = Re\, O(\omega^{2/3})$ for $t - t_n = O(\omega^{-2/3})$, $\omega \ll 1$. Therefore no simple scaling, like $\omega \ln \omega$ proposed by Mei, uniformly valid over the entire time domain can be found for the odd part of the drag in the low-frequency limit.

The results of calculations of the Oseen force from (3.1) are shown in figure 6 in comparison with the force given by (6.9) and Mei's numerical result. The asymptotic solution predicts well the even part and the odd part except the vicinity of the cuspidal points. A few terms of the Fourier transform approximates well the even part but are not sufficient to approximate the real behaviour of the odd part of the drag.

The flow field induced by the oscillating sphere in the far region, $r \gg \omega^{-1}$, also takes a form similar to (6.1). It follows from (6.8) that the relative displacement is

$\Delta z_p = \omega^{-1}(\sin\omega t - \sin\omega\tau) = O(\omega^{-1})$. The integrand of (2.3*a*) at $r \gg \omega^{-1}$ is not exponentially small only for a sufficiently large time interval, $\Delta\tau \sim r^{1/2} \gg \omega^{-1/2}$. For such values of $\Delta\tau$ we have $\Delta z_p^2 \Delta\tau^{-1} \ll 1$, and the expansion of the integrand (6.2) is valid in the far flow. Hence the velocity field can be presented as

$$u_z = h_1(r,t) + z^2 h_2(r,t) \quad \text{for} \quad r \gg \omega^{-1},$$

$$h_1 = \frac{3}{8\pi^{1/2}} \int_{-\infty}^{t} [2\Delta\tau\cos\omega\tau - \omega^{-1}(\sin\omega t - \sin\omega\tau)] \exp\left(-\frac{r^2}{4\Delta\tau}\right) \frac{d\tau}{\Delta\tau^{5/2}},$$
$$h_2 = \frac{3}{16\pi^{1/2}} \int_{-\infty}^{t} \omega^{-1}(\sin\omega t - \sin\omega\tau) \exp\left(-\frac{r^2}{4\Delta\tau}\right) \frac{d\tau}{\Delta\tau^{7/2}}.$$

The integrals including $\cos\omega\tau$ and $\sin\omega\tau$ are proportional to $\exp(-r\sqrt{2\omega})$ and hence are exponentially small for $r \gg \omega^{-1}$, $\omega \sim 1$. The dominant contribution comes from the terms proportional to $\sin\omega t$ so that

$$h_1 = -\frac{3}{2\omega r^3}\sin\omega t, \quad h_2 = \frac{9}{2\omega r^5}\sin\omega t.$$

As a result the velocity in the far field is

$$u_z = \frac{3\sin\omega t}{2\omega r^3}\left(-1 + \frac{3z^2}{r^2}\right) \quad \text{for} \quad r \gg \omega^{-1}.$$

Thus the velocity induced by the oscillating particle at distances large compared with a reverse frequency is also the potential dipole flow oscillating with time. Its strength equals the displacement amplitude ω^{-1} multiplied by the dimensionless flow rate 6π.

6.3. *Power-law dependent $V_p(t)$ with $m > 1/2$*

The condition of small-displacement motion (6.1) can be fulfilled not only for profiles with a finite displacement. It can be also satisfied if z_p grows slower than $t^{1/2}$. This takes place, for example, for the start-up power-law dependence. The particle and fluid are quiescent for $t < 0$, and for $t > 0$ the dimensional velocity is

$$V_p' = \gamma'(t')^{-m}.$$

The dimension of the constant γ' is $[lt^{m-1}]$. The characteristic velocity should be introduced as $U_c = \nu^{\lambda}\gamma'^{\kappa}$, with $\lambda = -m(1-2m)^{-1}$, $\kappa = (1-2m)^{-1}$. The dimensionless particle velocity and displacement are

$$V_p = t^{-m}, \quad z_p = \frac{t^{1-m}}{1-m}. \tag{6.10}$$

When $m < 1/2$ the particle displacement is larger than the diffusion length in the long-time limit, and this is the case of large-displacement motion. The Oseen force is close to the quasi-steady value decaying as t^{-2m} at $t \gg 1$. The small-displacement case occurs if $m > 1/2$. The index m should be less than unity; otherwise the displacement is singular as $t \to 0$.

The displacements scaled by l_D which enter into equations (6.4*a*) are

$$\zeta_p(t) = \frac{t^{1/2-m}}{2(1-m)}, \quad \Delta\zeta_p = t^{1/2-m}\Delta s_p,$$

where $\Delta s_p = (1 - s^{1-m})/2(1-m)$. As a result the velocity field for the power-law

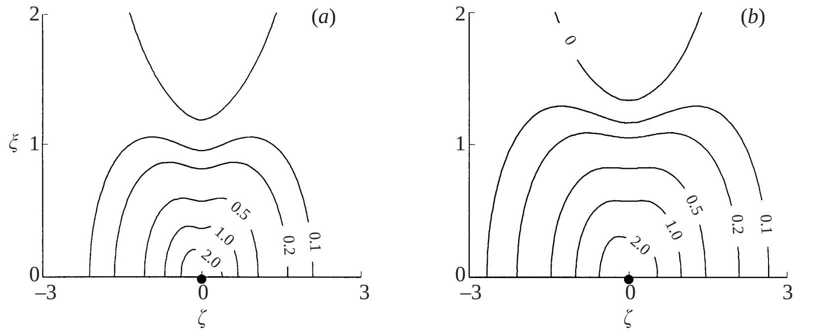

FIGURE 7. Disturbance velocity u_{zm} in the far field and long-time limit for the power-law slip-velocity profile $V_p = t^{-m}$, $t \gg 1$: (*a*) $m = 0.6$, (*b*) $m = 0.8$.

dependent $V_p(t)$ can be presented as

$$u_z = t^{-1/2-m}[g_{1m}(\eta) + \zeta^2 g_{2m}(\eta)] + O(t^{-2m}) \quad \text{for} \quad t \gg 1,\ \eta \sim 1,\ 1/2 < m < 1, \tag{6.11}$$

$$g_{1m} = \frac{3}{4\pi^{1/2}} \int_0^1 (s^{-m}\Delta s - \Delta s_p) \exp\left(-\frac{\eta^2}{\Delta s}\right) \frac{\mathrm{d}s}{\Delta s^{5/2}} + \frac{3}{2(1-m)} h(\eta),$$

$$g_{2m} = \frac{3}{2\pi^{1/2}} \int_0^1 \Delta s_p \exp\left(-\frac{\eta^2}{\Delta s}\right) \frac{\mathrm{d}s}{\Delta s^{7/2}} + \frac{3h'(\eta)}{2(1-m)\eta},$$

where $\Delta s = \Delta\tau/t = 1 - s$.

Figure 7 shows the results of calculations of the velocity field $u_{zm} = g_{1m}(\eta) + \zeta^2 g_{2m}(\eta)$ for various m. The streamwise velocity is symmetric in ζ. The magnitude of the disturbance is greater for greater m.

It should be stressed that the velocity in the far field for various small-displacement flow fields depends on the space coordinates in terms of the combination $\boldsymbol{\eta} = \boldsymbol{r}/l_D$. This means that the flow grows diffusively as $t^{1/2}$ in all directions. However the disturbance decay with time differs for the various slip-velocity profiles. This variation again can be explained by momentum conservation arguments. As the flow expands in all directions as $t^{1/2}$ the volume of the disturbed region is the same for all motion types, of order $(t^{1/2})^3 = t^{3/2}$. For the flow arising after a short-term motion the total momentum released into the fluid, $6\pi z_p(t_2)$, is constant. As a result we have in this case the disturbance velocity and the history force decaying as $t^{-3/2}$. For the start-up power-law velocity profile (6.10) the momentum released grows as t^{1-m}, and the disturbance decays as $t^{1-m}t^{-3/2} = t^{-1/2-m}$. The flow pattern for the self-similar solution with $V_p = \gamma t^{-1/2}$ is of a similar type at small γ and it decays as t^{-1}. Thus the self-similar solution yields the slowest decay with t in the far field.

The contribution of the different times to the velocity history integral (2.3*b*) is of the same order for all power laws of slip velocity considered. However for the small-displacement motions with $1/2 < m < 1$ only the time elapsed since the momentum was introduced into the fluid is significant while the translation of the momentum source is small in the long-time limit. Solution (6.3) describes this diffusion-dominated history flow.

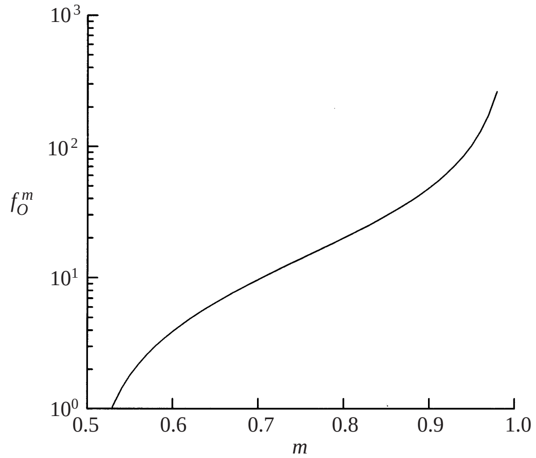

FIGURE 8. Oseen correction to the force on a particle for the power slip-velocity profile $V_p = t^{-m}$, $1/2 < m < 1$.

The unsteady Oseen force for the power dependence, in view of (3.2a), is given by

$$F_O^m = Re\, t^{-1/2-m} f_O^m,$$

$$f_O^m = \frac{9\pi^{1/2}}{4}\left\{\int_0^1 [(2s^{-m}-1)\Delta s - 2\Delta s_p]\frac{\mathrm{d}s}{\Delta s^{5/2}} - 2 - \frac{2}{3(1-m)}\right\}.$$

The history force dominates the quasi-steady Oseen force decaying as t^{-2m} at $t \gg 1$.

Figure 8 shows the Oseen correction as a function of m. The force grows rapidly with m. Note that the history force for the power-law slip velocity with $1/2 < m < 1$ is positive, as is that for the stopping motion (equation (6.7)) and unlike the self-similar solution with $m = 1/2$. The reason is that the particle translation is small for small-displacement motion, and the force is due to diffusion of positive momentum only.

7. Conclusion

The flow induced in the far region by a sphere which undergoes unsteady rectilinear motion with small Reynolds number and characteristic time comparable with the Oseen time scale is studied. New concise expressions for the disturbance field and the history force are derived. They make clearer the role of convection and diffusion of momentum for different time and length scales.

The types of particle unsteady motion are classified in terms of the ratio of the relative displacement to the diffusion length, $A = \Delta z_p/2\Delta\tau^{1/2}$. Different types having $A \gg 1$, $A \sim 1$ or $A \ll 1$ in the long-time limit are considered. For unsteady non-reversing motion with $A \gg 1$ for $\Delta\tau \gg 1$ (large-displacement motion) the flow past a sphere when expressed in terms of the elapsed time is the same as the steady-state laminar wake. The point source solution for the remainder of the far flow is also valid for the unsteady case.

When A is finite for all times, convection and the diffusion balance over the entire far region. The type of motion with velocity profile $V_p = \gamma t^{-1/2}$ for $t > 0$ satisfying this condition is studied. The self-similar solution, $\boldsymbol{u} = t^{-1}\boldsymbol{u}^{ss}(\boldsymbol{\eta})$, $\boldsymbol{\eta} = \boldsymbol{r}/2t^{1/2}$ is obtained for the disturbance flow. The history force is negative and inversely proportional to the time and differs for all times from the quasi-steady Oseen correction. The self-similar flow can be a good test of numerical simulations of unsteady flows past a small sphere.

The general form of velocity field, $u_z = g_1(\eta, t) + \zeta^2 g_2(\eta, t)$, $\zeta = z/2t^{1/2}$, for $t \gg 1$, is derived for small-displacement motion. Several examples of the particle velocity profile having $A \ll 1$ in the long-time limit are considered: motion during a small time interval; an oscillating sphere; start-up motion with power-law slip velocity $V_p(t) = t^{-m}$, $1/2 < m < 1$. The history force is positive for the first and third profiles.

The present results can be generalized for rectilinear motion of an axisymmetric body translating at small Reynolds number along its axis of symmetry. For the above analysis to be valid the leading-order force on the body must be aligned with the direction of the motion and must depend linearly on the slip velocity, $\boldsymbol{F} = -6\pi \boldsymbol{\Phi} \cdot \boldsymbol{V}_p$. Here $6\pi\boldsymbol{\Phi}$ is the Oseen resistance tensor. $\boldsymbol{\Phi}$ is a unit tensor for the spherical particle. To derive the expressions for the disturbance field and the history force for the axisymmetric body one should multiply the right-hand side of equation (1.1) and all subsequent equations by Φ_{zz}.

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